Notes on the new low-pass filter for the orography field

Internal report
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March 23, 2023

Abstract

These notes are meant to describe the properties of a new orography filter/smoother, recently implemented as an option in the GenPhysX generator.

1 Brief review on filters and response functions

1.1 A simple, generic 1-d filter

Consider a 1-dimensional uniform grid, with resolution Δx and grid points labeled by integers,

$$x_n = n\Delta x \ , \ n = 0, 1, 2, \cdots \tag{1}$$

Now consider a field H (for example, GEM's orography field "ME"") defined on the same grid and represented by

$$H(x_n) = H_n \tag{2}$$

The action of a typical filter/smoother acting on the field H can be described by a linear operator of the form

$$H_n \to H_n^F = c_1 H_n + \sum_{m=1}^{p-1} (c^r_{m+1} H_{n+m} + c^l_{m+1} H_{n-m}) \tag{3}$$

where H^F is the filtered field. Note that the filter can be seen as a linear combination of $(2p-1)$ neighboring values, i.e. the original "central" value combined with those of $(p-1)$ "neighbors to the right" and $(p-1)$ "neighbors to the left". Assuming left-right symmetry, i.e. choosing $c^r_m = c^l_m = c_m$, then that the filter is defined by a total of p coefficients, $c1, \cdots, c_p$,

$$H_n \to H_n^F = c_1 H_n + \sum_{m=1}^{p-1} c_{m+1} (H_{n+m} + H_{n-m}) \tag{4}$$

1.2 Response function R

To describe the impact of a filter on individual spatial scales, let us consider the Fourier decomposition of field H in terms of wavenumbers. For a given scale or wavelength L, let

$$K = \frac{2\pi}{L} = \text{wavenumber} \tag{5}$$

and

$$k = \frac{\Delta x}{L} = \text{non-dimensional wavenumber} \tag{6}$$

such that

$$K x_n = \frac{2\pi}{L} n \Delta x = 2\pi k n \tag{7}$$

The minimum wavelength resolved by the grid is $L_{min} = 2\Delta x$, so that the range of the non-dimensional wavenumber is $-1/2 \leq k \leq 1/2$. Therefore the complex Fourier decomposition of H at the grid-point n, written in terms of the non-dimensional wavenumber k, reads

$$H_n = \int_{-1/2}^{1/2} A(k) e^{i2\pi kn}\, dk \tag{8}$$

where $A(k)$ provides the Fourier coefficients of $H(x)$.

Using the filter operator in eq. 4, we can relate the Fourier coefficients $A(k)$ of the original field H to the coefficients $A^F(k)$ of the filtered field H^F, as follows:

$$\begin{aligned}
H_n^F &= c_1 H_n + \sum_{m=1}^{p-1} c_{m+1}(H_{n+m} + H_{n-m}) \\
&= c_1 \int_{-1/2}^{1/2} A(k) e^{i2\pi kn}\, dk + \sum_{m=1}^{p-1} c_{m+1} \int_{-1/2}^{1/2} A(k) e^{i2\pi kn} \left(e^{i2\pi km} + e^{-i2\pi km}\right) dk \\
&= \int_{-1/2}^{1/2} \left\{ \left[c_1 + 2\sum_{m=1}^{p-1} c_{m+1} \cos(2\pi km) \right] A(k) \right\} e^{i2\pi kn}\, dk \\
&= \int_{-1/2}^{1/2} A^F(k) e^{i2\pi kn}\, dk \\
\Longrightarrow A^F(k) &= \left[c_1 + 2\sum_{m=1}^{p-1} c_{m+1} \cos(2\pi km) \right] A(k) \qquad (9)
\end{aligned}$$

In terms of wavenumbers, the filter's **response function** R [1] is here defined as the ratio of Fourier coefficients A^F/A,

$$R(k) = \frac{A^F(k)}{A(k)} = c_1 + 2\sum_{m=1}^{p-1} c_{m+1} \cos(2\pi km) \tag{10}$$

[1]Often one computes the ratio of power spectra – i.e. the power spectrum of the filtered field, divided by the power spectrum of the unfiltered one – as an estimate of the response function. Since the power spectrum is basically $P(k) \sim |A(k)|^2$, the ratio of power spectra is not really R; it is actually given by its square, $\frac{P^F(k)}{P(k)} = (R(k))^2$.

For a given non-dimensional wavenumber k, or equivalently for a given scale $L = \Delta x/k$, the response function $R(k)$ measures how much the amplitude of the original field (in our case, the orography) is amplified or damped by the filter. Notice that the coefficients c_n appear on the r.h.s. as the coefficients of a truncated Fourier expansion, i.e. the truncated Fourier/cosine expansion of $R(k)$.

1.3 Impact of the filter on the spatial average of the field

Assuming a periodic grid (or, in the non-periodic case, neglecting boundary terms), the impact of the filter on the mean field is dictated by the sum of the filter coefficients,

$$\begin{aligned} \left\langle H^F \right\rangle &= c_1 \langle H \rangle + \sum_{m=1}^{p-1} c_{m+1}(\langle H \rangle + \langle H \rangle) \\ &= \left[c_1 + 2 \sum_{m=1}^{p-1} c_{m+1} \right] \langle H \rangle \\ \Longrightarrow \frac{\left\langle H^F \right\rangle}{\langle H \rangle} &= c_1 + 2 \sum_{m=1}^{p-1} c_{m+1} \end{aligned} \tag{11}$$

where $\langle \cdots \rangle$ indicates the operation of spatial-averaging. Notice that the sum of coefficients on the r.h.s of eq. 11 is the same as the response function at the largest scale (i.e. when $L \to \infty$, or equivalently when $k \to 0$),

$$\lim_{k \to 0} R(k) = c_1 + 2 \sum_{m=1}^{p-1} c_{m+1} \tag{12}$$

If we want the filter to preserve the average value of the field (i.e. if we want to impose the condition $\left\langle H^F \right\rangle = \langle H \rangle$), we may normalize the coefficients c_m such that

$$c_1 + 2 \sum_{m=1}^{p-1} c_{m+1} = 1 \tag{13}$$

The results presented in section 1 apply to most 1-d filters. In the following sections, we apply these results to 2 specific filter, the "old" 2dx filter and the "new" low-pass filter.

2 Design and properties of the old 2dx filter

This filter involves only the nearest pair of neighboring grid points, and can be written as follows,

$$H_n^F = (1 - f)H_n + \frac{f}{2}(H_{n+1} + H_{n-1}) \tag{14}$$

Notice that it depends on only one parameter f, which is proportional to the relative weight $(f/2)$ given to the neighboring values. Notice also that the coefficients/weights are already normalized, i.e. $(1-f)+f/2+f/2=1$, so that the application of this filter will preserve the mean.

Using eq.10, we derive the corresponding response function,

$$\begin{aligned} R_f(k) &= (1-f)+2\cdot\frac{f}{2}\cos(2\pi k) \\ &= 1-f\left[1-\cos(2\pi k)\right] \\ &= 1-2f\sin^2(\pi k) \end{aligned} \tag{15}$$

For the scale/wavelength of size 2-dx, i.e. for the largest wavenumbers $k=\pm 1/2$, the response function is

$$R_f(\pm 1/2) = 1-2f\sin^2(\pm\pi/2) = 1-2f \tag{16}$$

In particular, if we choose the parameter $f=1/2$, we get

$$R_{1/2}(\pm 1/2) = 0 \tag{17}$$

This choice (i.e. with $f=1/2$) corresponds to the what we call the "old 2dx filter". The name "2dx filter" comes from the fact it can completely eliminate/filter-out the 2dx scales, which was probably its original objective. The problem with this filter is that it strongly impacts scales much larger than 2dx as well – see fig.1. Therefore the interest in trying alternative, sharper filters.

3 Design and properties of the new low-pass filter

Consider an idealized low-pass filter that eliminates all scales smaller than a given cut-off scale $L_c = r_c\Delta x$, while keeping the larger scales intact. This can be represented by the "ideal" response function

$$R_\infty(k) = \begin{cases} 1, & 0<|k|<\frac{1}{r_c} \\ 0, & \frac{1}{r_c}<|k|<\frac{1}{2} \end{cases} \tag{18}$$

(The meaning of the index ∞ in R_∞ will be explained later).

Let us consider the Fourier decomposition of this response function $R(k)$. We do this because the coefficients of the Fourier decomposition of R will eventually lead us to the coefficients of the digital filter in real space – see eq.10.

For an even response function R of k, i.e. assuming $R(-k)=R(k)$, its general cosine-decomposition reads

$$R(k) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(2\pi kn) \tag{19}$$

where the expansion coeffiecients are

$$\begin{aligned} a_n &\equiv 2\int_{-1/2}^{1/2} R(k)\cos(2\pi kn)\,dk \\ &= 4\int_0^{1/2} R(k)\cos(2\pi kn)\,dk \quad ,\ n=0,1,2,\cdots \end{aligned} \tag{20}$$

In the case of the idealized filter above, we have

$$\begin{aligned} a_n &= 4\int_0^{1/2} R_\infty(k)\cos(2\pi kn)\,dk \\ &= 4\int_0^{1/r_c}\cos(2\pi kn)\,dk \\ &= \frac{2}{\pi n}\sin(2\pi n/r_c)\,, \quad n=1,2,\cdots \end{aligned} \tag{21}$$

and

$$a_0 = 4\int_0^{1/2} R_\infty(k)\,dk = 4\int_0^{1/r_c} dk = \frac{4}{r_c} \tag{22}$$

that is, the idealized response function can be expanded as

$$\begin{aligned} R_\infty(k) &= \frac{2}{r_c} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin(2\pi n/r_c)\cos(2\pi kn) \\ &= \frac{2}{r_c}\left\{1+2\sum_{n=1}^{\infty}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\right]\cos(2\pi kn)\right\} \end{aligned} \tag{23}$$

Consider now the corresponding truncated series of p terms,

$$R_p(k) = \frac{2}{r_c}\left\{1+2\sum_{n=1}^{p-1}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\right]\cos(2\pi kn)\right\} \tag{24}$$

of which the idealized filter is a limiting case, i.e.

$$R_\infty(k) = \lim_{p\to\infty} R_p(k) \tag{25}$$

This truncated series is, of course, an approximation of the ideal filter. This type of approximation often leads to ripples and overshoots. To reduce somewhat the amplitude of the ripples and smooth the truncated filter, let us perform a rectangular-window averaging [2] around each wavenumber k, with an avering window of $[k-1/p, k+1/p]$, i.e.

$$\begin{aligned} \hat{R}_p(k) &= \frac{p}{2}\int_{k-1/p}^{k+1/p} R_p(\nu)\,d\nu \\ &= \frac{p}{2}\int_{k-1/p}^{k+1/p}\frac{2}{r_c}\left\{1+2\sum_{n=1}^{p-1}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\right]\cos(2\pi\nu n)\right\}\,d\nu \\ &= \frac{2}{r_c}\left\{1+2\sum_{n=1}^{p-1}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\right]\cdot\frac{p}{2}\int_{k-1/p}^{k+1/p}\cos(2\pi\nu n)\,d\nu\right\} \\ &= \frac{2}{r_c}\left\{1+2\sum_{n=1}^{p-1}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\right]\cdot\frac{p}{2}\left[\frac{\sin(2\pi n(k+1/p))-\sin(2\pi n(k-1/p))}{2\pi n}\right]\right\} \\ &= \frac{2}{r_c}\left\{1+2\sum_{n=1}^{p-1}\left[\frac{\sin(2\pi n/r_c)}{2\pi n/r_c}\cdot\frac{\sin(2\pi n/p)}{2\pi n/p}\right]\cos(2\pi kn)\right\} \end{aligned} \tag{26}$$

[2] This is equivalent to perfoming a convolution with a top-hat function of width $2/p$, as explained in a separate note.

where we used the trigonometric identity $\sin(a+b) - \sin(a-b) = 2\cos(a)\sin(b)$. This shows that the effect of the rectangular-window averaging simply amounts to multiplying the original coefficients of the truncated Fourier expansion by the so-called sigma factors [3],

$$\sigma_{np} = \frac{\sin(2\pi n/p)}{2\pi n/p} \tag{27}$$

where p is the truncation number.

Comparing the expansion of $\hat{R}_p(k)$ with equation (10), we may inticipate that $\hat{R}_p(k)$ is the response function of the following filter:

$$H_n^F = c_1 H_n + \sum_{m=1}^{p-1} c_{m+1}(H_{n+m} + H_{n-m}) \tag{28}$$

with

$$c_1 = \frac{2}{r_c} \tag{29}$$

and

$$c_{m+1} = \frac{2}{r_c}\left[\frac{\sin(2\pi m/r_c)}{2\pi m/r_c} \cdot \frac{\sin(2\pi m/p)}{2\pi m/p}\right] \ , \quad m = 1, \cdots, p-1 \tag{30}$$

Finally, one may want normalize these coefficients in order to preserve the mean. In this case, it suffices to re-scale them as follows,

$$c_m \to \hat{c}_m = \frac{c_m}{S_p} \tag{31}$$

where

$$S_p = c_1 + 2\sum_{m=1}^{p-1} c_{m+1} \tag{32}$$

is the sum of the non-normalized coefficients of the truncated filter. In sum, the new filter is defined by the set of p normalized coefficients $\hat{c}_n$.

Notice that these coefficients depend on 2 integer parameters, (i) the cut-off scale parameter r_c, which indicates the threshold wavelength (as a multiple of Δx) beyond which the amplitude of the signal should be reduced – see figure 1; and (ii) the truncation parameter p, which basically controls the sharpness of the filter: the larger the value of p, the sharper the filter – see figure 2.

In practice, it is recommended to choose $p > r_c$ (i.e. an averaging-window larger than the wavelength we expect to filter out), and possibly $p >> r_c$, in which case the $r_c\Delta x$ becomes the wavelength whose amplitude is reduced by a factor of 0.5 – see figure 1.

[3]Note that these sigma factors may be expressed in terms of the so-called **normalized sinc function**, defined as $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. This is also discussed in the additional notes.

4 2-dimensional extension

Take the normalized coefficient $\hat{c}_n$ defined above. Now, assuming a 2-d cartesian grid with uniform resolution and gridpoints labeled as $(x_m, y_n) = (m\Delta x, n\Delta y)$, and a 2-d field $H(x_m, y_n) = H_{m,n}$, the filter can be implemented as a sequential application of the 1-d filter, e.g. first in the x-direction followed by an application in the y-direction (note that this sequence is commutable), i.e.

$$W_{m,n} = \hat{c}_1 H_{m,n} + \sum_{i=1}^{p-1} \hat{c}_{i+1}(H_{m+i,n} + H_{m-i,n}) \tag{33}$$

$$H^F_{m,n} = \hat{c}_1 W_{m,n} + \sum_{j=1}^{p-1} \hat{c}_{j+1}(W_{m,n+j} + W_{m,n-j}) \tag{34}$$

where $W_{m,n}$ is simply an auxiliary field, resulting from the intermediate filtering in the x-direction. Examples of power spectra and response functions for 2-d global, filtered orography fields are shown in figures 3 and 4.

5 Implementation of the new low-pass filter in GenPhysX, including extra options

In GenphysX, the values of r_c and p are controled by the following parameters

LPASSFLT_RC_DELTAX (default = 3.0)
LPASSFLT_P (default = 20)

Two additional options are also available, as described below. A separate document ("*Additional notes on the properties of the low-pass filter*") provides more details on these parameters and options.

Important:

- As already mentioned, it is preferable that the parameters p and r_c satisfy the constraint $p > r_c$. In grid-point space this constraint translates into an intuitive condition: "*one should use at least* $2\,r_c$ *(*$\sim 2p$*) gridpoints to represent (and eventually filter out) a wave of length* $r_c \Delta x$". In fact, **the latest recommendation is to set** $p = 5\,r_c$. More details are available in the additional notes.

- The application of the options described below (mask and local-min-max conditions) has an impact on the filter response. That type of impact is also discussed in the additional notes.

5.1 Optional masking field

It was observed that the new filter could still generate some small noisy ripples over the oceans near the coast lines. To remove those noisy values, one may activate a "mask" operator based on the properties of a chosen field, e.g. by taking H^F only over cells where the orography variance – the field SSS – is large enough (e.g. only where $SSS > 0.01m$), and keeping $H_F = H$ otherwise. This masking operation is available and is controled by the following settings in GenPhysX:

LPASSFLT_MASK_OPERATOR = 1 (default is 0, which means "don't apply it"; 1 means "larger than"; and -1 means "lesser than" the threshold defined below)

LPASSFLT_MASK_THRESHOLD = 0.01 (in meters; default value is 100.)

5.2 Optional local-min-max constraint

One may also impose that the filter preserve local minima/maxima, for instance by defining two auxiliary fields,

$$H^{lmin}_{m,n} = \text{minimum value among the 9 neighboring points around } H_{m,n} \qquad (35)$$

$$H^{lmax}_{m,n} = \text{maximum value among the 9 neighboring points around } H_{m,n} \qquad (36)$$

and imposing

$$H^F_{m,n} = \min(\max(H^F_{m,n}, H^{lmin}_{m,n}), H^{lmax}_{m,n}) \qquad (37)$$

In GenPhysX, this option is activated by the logical switch

LPASSFLT_APPLY_MINMAX = True (default is True)

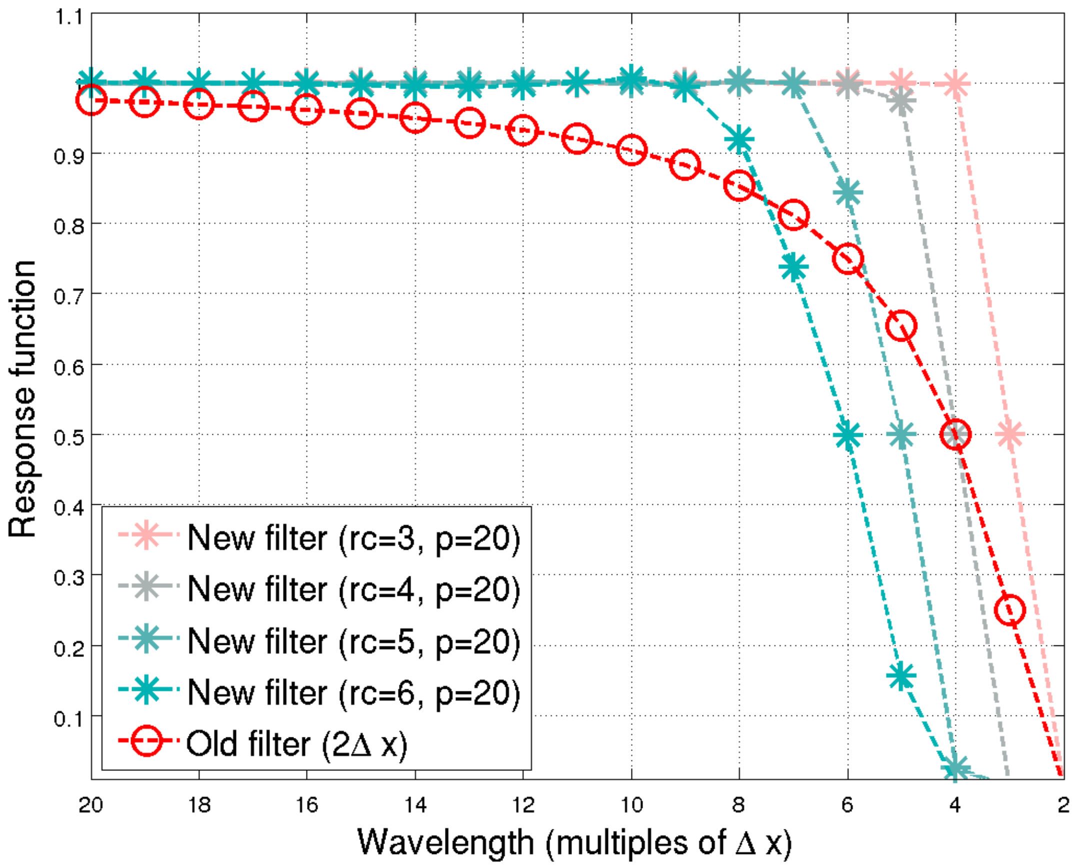

Figure 1: Sensitivity to cut-off parameter r_c.

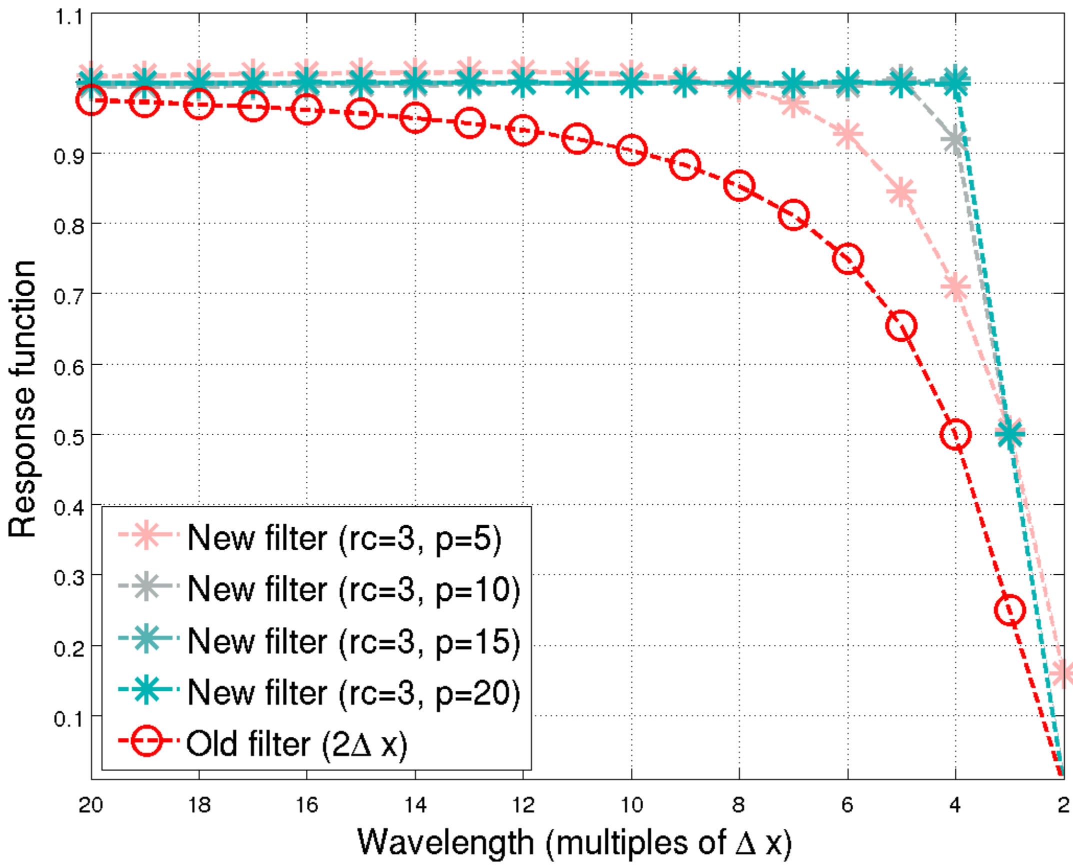

Figure 2: Sensitivity to sharpness parameter p.

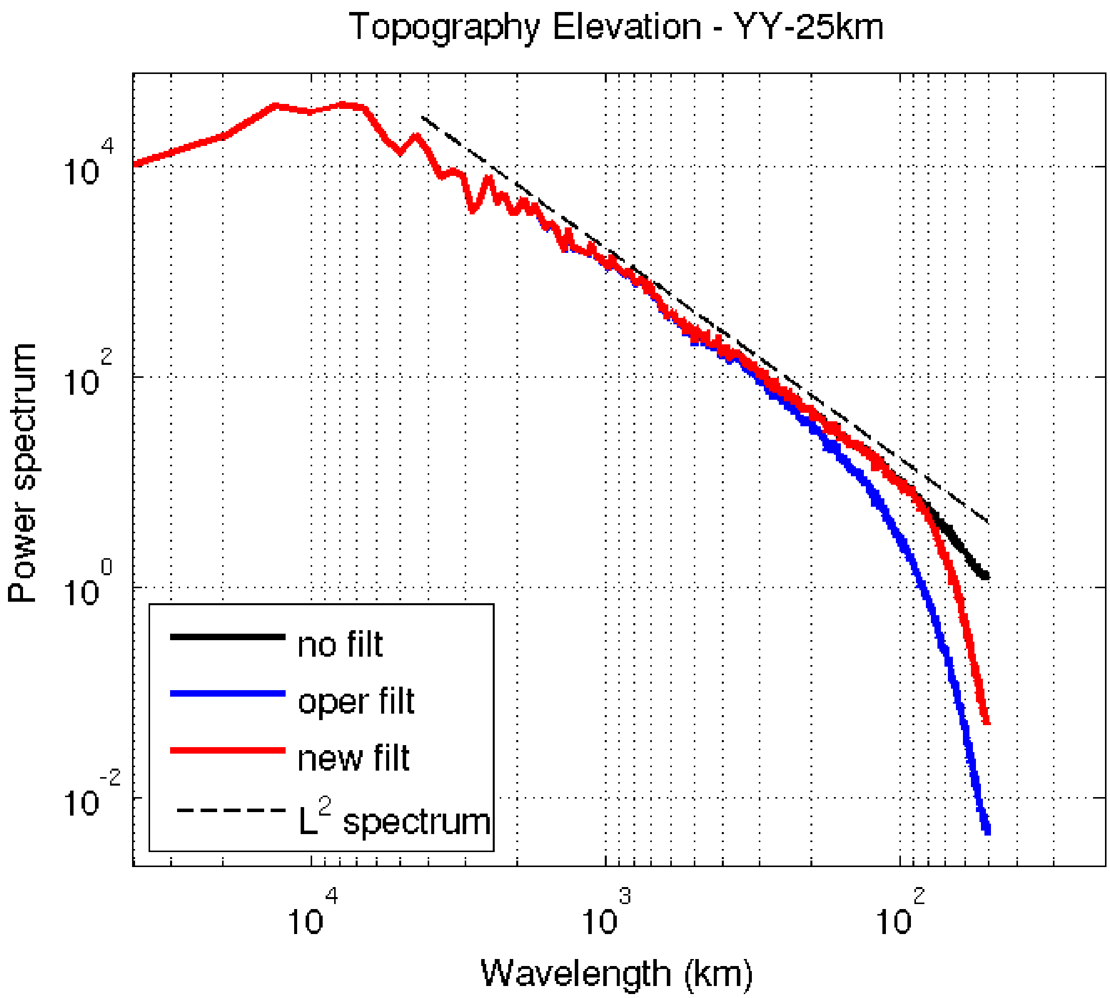

Figure 3: Power spectra of the orography field ME for a global Yin-Yang grid with $\Delta x \approx 25$ km: unfiltered field (black); smoothed by the operational 2dx filter (blue); smoothed by the new low-pass filter with $r_c = 3$ and $p = 20$ (red); and a reference L^2 spectrum (dashed).

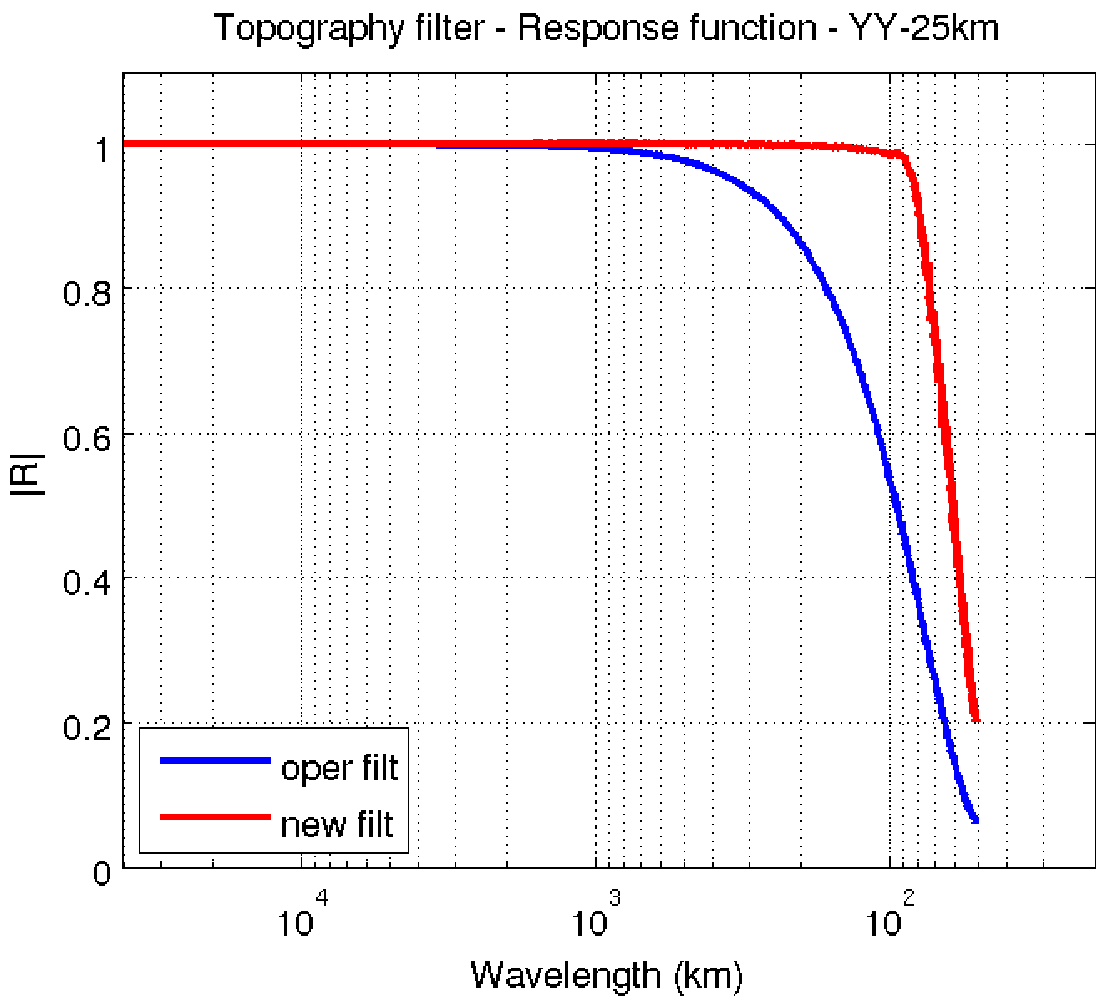

Figure 4: Response function of the 2dx filter and the new low-pass filter, based on the spectra shown in figure 3.