Notes on the orography variance and slope covariances: filling gaps and separating scales

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Abstract

"Separating scales" from databases is recommended when calculating subgrid orographic fields for physical parametrizations. In principle, this could be achieved by pre-pocessing the databases themselves. Unfortunately, these pre-processed databases are not yet available at CMC. These notes are meant to provide an alternative to this pre-processing, by describing a method to approximately (i) fill up the high-wavenumber spectrum of orography and (ii) separate scales of the subgrid variance and slope covariances in a way that is convenient for the boundary layer (PBL) and subgrid orography (SSO) parameterizations.

1 Total variance of subgrid orography

Suppose that the orography power spectrum $S(K)$ generally obeys a power law,

$$S(k) = A \cdot K^{-\beta} \tag{1}$$

where A is some amplitude (which varies from a grid-cell to another), K is the total wavenumber, and β is a known exponent. In fact, observations suggest that $\beta \sim 2$ (see Fig. 1). Then the total variance of the sugbrid orography for a given grid-cell would be

$${\sigma_t}^2 = \int_{K_m}^{\infty} S(K)\, dK = \frac{A}{(\beta - 1)} {K_m}^{-(\beta-1)} \tag{2}$$

where

$$K_m = \frac{2\pi}{L_m} \tag{3}$$

is the wavenumber associated with the grid-cell size, which is given by the model resolution (i.e. $L_m \sim \sqrt{\Delta x\, \Delta y}$).

2 Total variance reconstruction from partial variance

An estimate of orography variance is currently produced by the generator of geophysical fields (e.g. Genesis or GenPhysX) for each grid-cell, using elevation data from a database chosen by the user.

Suppose that the chosen database has a resolution L_b (e.g. GTOPO30 has $L_b \sim 900m$), that $L_b < L_m$ (i.e. the resolution of database is higher than the model resolution), and let

$$K_b = \frac{2\pi}{L_b} \tag{4}$$

be the associated wavenumber. Then the grid-cell orography variance obtained from this database would be

$${\sigma_b}^2 = \int_{K_m}^{K_b} S(k)\, dk = \frac{A}{(\beta-1)} \left({K_m}^{-(\beta-1)} - {K_b}^{-(\beta-1)} \right) \tag{5}$$

Note that σ_b is an underestimation of the total variance, since it lacks contributions from the small scales not resolved by the database.

Still, once σ_b is obtained, we may find (and eliminate) the unknown amplitude A, and so reconstruct the total variance σ_t from σ_b and from the resolution ratio:

$${\sigma_t}^2 = {\sigma_b}^2 \frac{{K_m}^{-(\beta-1)}}{\left({K_m}^{-(\beta-1)} - {K_b}^{-(\beta-1)} \right)} = \frac{{\sigma_b}^2}{1 - \left(\frac{L_b}{L_m} \right)^{\beta-1}} \tag{6}$$

Defining the auxiliary parameter

$$w_{mb} = \left(\frac{L_b}{L_m} \right)^{\beta-1} \tag{7}$$

we have

$${\sigma_t}^2 = \frac{{\sigma_b}^2}{(1 - w_{mb})} \tag{8}$$

Note 1: The above equations only make sense if the database has enough resolution compared to your model resolution, i.e. if $L_b < L_m$. Ideally, you should have $L_b \ll L_m$. If not, just get yourself a better database!

3 Separation of total variance

Now suppose that we want to separate the total variance σ_t into a *large*- and a *small*-component, based on a prescribed separation scale L_s (and the associated wavenumber $K_s = 2\pi/L_s$). This type of separation is in fact useful for the partition of orography forcing between the PBL scheme and the SSO scheme, in which case the recommended separation scale is $L_s \sim 5km$.

Assuming that $L_b < L_s < L_m$, then we may decompose

$$\sigma_t{}^2 = \sigma_{large}{}^2 + \sigma_{small}{}^2 \tag{9}$$

where

$$\sigma_{small}{}^2 = \int_{K_s}^{\infty} S(K)\, dK = \frac{A}{(\beta-1)} K_s{}^{-(\beta-1)} = \sigma_t{}^2 \left(\frac{K_s}{K_m}\right)^{-(\beta-1)} = \sigma_t{}^2 \left(\frac{L_s}{L_m}\right)^{\beta-1} \tag{10}$$

and

$$\sigma_{large}{}^2 = \int_{K_m}^{K_s} S(K)\, dK = \sigma_t{}^2 \left[1 - \left(\frac{L_s}{L_m}\right)^{\beta-1}\right] \tag{11}$$

Note that the above separation formula may be generalized to the case when $L_m < L_s$ (e.g. high-resolution models), if we define the auxiliary separation parameter

$$w_{ms} = \left[\min\left(\frac{L_s}{L_m}, 1\right)\right]^{\beta-1} \tag{12}$$

so that

$$\sigma_{small}{}^2 = w_{ms} \cdot \sigma_t{}^2 \tag{13}$$

$$\sigma_{large}{}^2 = (1 - w_{ms}) \cdot \sigma_t{}^2 \tag{14}$$

If your model resolution is higher than the separation scale, i.e. $L_m < L_s$, then $w \to 1$ and all the variance goes into the *small*-scale component.

Combining the above relations with those of the previous section, we finally get

$$\sigma_{small}{}^2 = \frac{w_{ms}}{(1 - w_{mb})} \cdot \sigma_b{}^2 \tag{15}$$

$$\sigma_{large}{}^2 = \frac{(1 - w_{ms})}{(1 - w_{mb})} \cdot \sigma_b{}^2 \tag{16}$$

which show how to estimate the requested output (i.e. $\sigma_{small}{}^2$ and $\sigma_{large}{}^2$) from the provided input (i.e. $\sigma_b{}^2$ and the resolution ratios).

Note 2: We should probably consider the notion of *effective* resolution, both for the model and for the database, when defining the values of the parameters L_m and L_b.

4 Slope covariances

A similar re-scaling may be derived for the slope covariances,

$$G_{xx} = \int_{K_m}^{K_s} k^2 S(K)\, dK \tag{17}$$

$$G_{yy} = \int_{K_m}^{K_s} l^2 S(K)\, dK \tag{18}$$

$$G_{xy} = \int_{K_m}^{K_s} kl S(K)\, dK \tag{19}$$

where k and l indicate wavenumbers in the x- and y- directions respectively, and $K^2 = k^2 + l^2$. Note that the SSO scheme only needs the *large*-scale component of these quantities, therefore the integral limits K_m to K_s.

Assuming approximate isotropy, i.e. that $G_{xx} \sim G_{yy} \sim 0.5 \int (k^2 + l^2) S(K)\, dK$, we could estimate

$$G_{xx} \sim 0.5 \int_{K_m}^{K_s} K^2 S(K)\, dK = 0.5 \frac{A}{(3-\beta)} \left({K_s}^{(3-\beta)} - {K_m}^{(3-\beta)} \right) \tag{20}$$

If we use all the wavelengths available in the high-resolution database to compute the covariance, we will be oversestimating this quantity by

$$G^b_{xx} \sim 0.5 \int_{K_m}^{K_b} K^2 S(K)\, dK = 0.5 \frac{A}{(3-\beta)} \left({K_b}^{(3-\beta)} - {K_m}^{(3-\beta)} \right) \sim \frac{1}{r} G_{xx} \tag{21}$$

where the factor r is given by

$$r = \frac{\left({K_s}^{(3-\beta)} - {K_m}^{(3-\beta)} \right)}{\left({K_b}^{(3-\beta)} - {K_m}^{(3-\beta)} \right)} = \frac{\left(\frac{L_m}{L_s} \right)^{(3-\beta)} - 1}{\left(\frac{L_m}{L_b} \right)^{(3-\beta)} - 1} \tag{22}$$

Note that, if $\beta \sim 2$, then $0 < r < 1$ whenever $L_b < L_s < L_m$, consistent with the statement that G^b_{xx} is an overestimate of G_{xx}.

Assuming that G^b_{xx} is the quantity provided by GenPhysX, then the adjusted (re-scaled) value we actually want would be

$$G_{xx} \sim r \cdot G^b_{xx} \tag{23}$$

We could use the same re-scaling to adjust G_{yy} and G_{xy}.

Note 3: Alternatively (ideally), we would avoid the need of such re-scaling if we could compute the covariances directly from a database of resolution $L_b \sim L_s$.

5 Generalization to the case of a 2-exponent spectrum

Some studies suggest that the orography spectrum may actually have the form:

$$S(K) = A \cdot \begin{cases} K^{-\beta_1}, & \text{if } K \le K_0 \\ K_0^{(\beta_2-\beta_1)} K^{-\beta_2}, & \text{if } K > K_0 \end{cases} \tag{24}$$

where β_1 and β_2 are distinct exponents, and K_0 is a the wavenumber where the that change occurs. For instance, Beljaars et al. 2004 propose $\beta_1 = 1.9$, $\beta_2 = 2.8$, and $K_0 = 0.003m^{-1}$ (i.e. $L_0 \sim 2km$).

Hereafter we will assume that the database has a sufficiently high resolution, i.e. that $L_b < L_0$ and $L_b < L_m$. In this case, the calculation of the total variance splits into 2 cases:

(i) if $L_m > L_0$ ($K_m < K_0$):

$$\begin{aligned} {\sigma_t}^2 &= \int_{K_m}^{\infty} S(K)\,dK & (25) \\ &= A \cdot \left[\frac{1}{(\beta_1 - 1)} {K_m}^{-(\beta_1-1)} + \frac{(\beta_1 - \beta_2)}{(\beta_1 - 1)(\beta_2 - 1)} {K_0}^{-(\beta_1-1)} \right] & (26) \end{aligned}$$

(ii) if $L_m < L_0$ ($K_m > K_0$):

$${\sigma_t}^2 = \int_{K_m}^{\infty} S(K)\,dK = A \cdot \left[\frac{K_0^{(\beta 2-\beta 1)}}{(\beta_2 - 1)} {K_m}^{-(\beta_2-1)} \right] \tag{27}$$

Meanwhile, the calculation of ${\sigma_b}^2$ also falls into 2 cases:

(i) if $L_m > L_0$ ($K_m < K_0$):

$$\begin{aligned} {\sigma_b}^2 &= \int_{K_m}^{K_b} S(K)\,dK & (28) \\ &= A \cdot \left[\frac{1}{(\beta_1 - 1)} {K_m}^{-(\beta_1-1)} + \frac{(\beta_1 - \beta_2)}{(\beta_1 - 1)(\beta_2 - 1)} {K_0}^{-(\beta_1-1)} - \frac{K_0^{(\beta 2-\beta 1)}}{(\beta_2 - 1)} {K_b}^{-(\beta_2-1)} \right] & (29) \\ &= {\sigma_t}^2 \cdot \left[1 - \frac{1}{c} \left(\frac{L_b}{L_m} \right)^{(\beta_2-1)} \right] & (30) \end{aligned}$$

where

$$c = \frac{(\beta_2 - 1)}{(\beta_1 - 1)} \left(\frac{L_0}{L_m} \right)^{(\beta_2-\beta_1)} + \frac{(\beta_1 - \beta_2)}{(\beta_1 - 1)} \left(\frac{L_0}{L_m} \right)^{(\beta_2-1)} \tag{31}$$

(ii) if $L_m < L_0$ ($K_m > K_0$):

$$\begin{aligned} {\sigma_b}^2 &= \int_{K_m}^{K_b} S(K)\,dK = A \cdot \left[\frac{K_0^{(\beta 2-\beta 1)}}{(\beta_2 - 1)} \left({K_m}^{-(\beta_2-1)} - {K_b}^{-(\beta_2-1)} \right) \right] & (32) \\ &= {\sigma_t}^2 \cdot \left[1 - \left(\frac{L_b}{L_m} \right)^{(\beta_2-1)} \right] & (33) \end{aligned}$$

Note that the 2 cases may now be unified as follows:

$$\sigma_t{}^2 = \frac{\sigma_b{}^2}{(1 - w_{mb})} \tag{34}$$

where

$$w_{mb} = \frac{1}{c}\left(\frac{L_b}{L_m}\right)^{(\beta_2 - 1)} \tag{35}$$

$$c = \frac{(\beta_2 - 1)}{(\beta_1 - 1)} a^{(\beta_2 - \beta_1)} + \frac{(\beta_1 - \beta_2)}{(\beta_1 - 1)} a^{(\beta_2 - 1)} \tag{36}$$

$$a = \min\left(\frac{L_0}{L_m}, 1\right) \tag{37}$$

As for the separation of scales, we must consider 2 different cases:
(i) if $L_m > L_s$ ($K_m < K_s$), i.e. for a relatively course-resolution model:

$$\begin{aligned} \sigma^2_{small} &= \int_{K_s}^{\infty} S(K)\, dK & (38) \\ &= A \cdot \left[\frac{1}{(\beta_1 - 1)} K_s{}^{-(\beta_1 - 1)} + \frac{(\beta_1 - \beta_2)}{(\beta_1 - 1)(\beta_2 - 1)} K_0{}^{-(\beta_1 - 1)}\right] & (39) \\ &= \sigma_t^2 \cdot \left(\frac{L_s}{L_m}\right)^{(\beta_1 - 1)} \frac{\left[1 + \frac{(\beta_1 - \beta_2)}{(\beta_2 - 1)}\left(\frac{L_0}{L_s}\right)^{(\beta_1 - 1)}\right]}{\left[1 + \frac{(\beta_1 - \beta_2)}{(\beta_2 - 1)}\left(\frac{L_0}{L_s}\right)^{(\beta_1 - 1)}\left(\frac{L_s}{L_m}\right)^{(\beta_1 - 1)}\right]} & (40) \end{aligned}$$

(ii) if $L_m < L_s$ ($K_m > K_s$), i.e. for a relatively high-resolution model:

$$\sigma^2_{small} = \sigma_t^2 \tag{41}$$

The relations above may also be unified as follows:

$$\sigma^2_{small} = w_{ms} \cdot \sigma_t^2 \tag{42}$$

where

$$w_{ms} = b^{(\beta_1 - 1)} \frac{\left[1 + \frac{(\beta_1 - \beta_2)}{(\beta_2 - 1)}\left(\frac{L_0}{L_s}\right)^{(\beta_1 - 1)}\right]}{\left[1 + \frac{(\beta_1 - \beta_2)}{(\beta_2 - 1)}\left(\frac{L_0}{L_s}\right)^{(\beta_1 - 1)} b^{(\beta_1 - 1)}\right]} \tag{43}$$

and

$$b = \min\left(\frac{L_s}{L_m}, 1\right) \tag{44}$$

In sum, the relations

$$\sigma^2_{small} = \frac{w_{ms}}{(1 - w_{mb})} \cdot \sigma^2_b \tag{45}$$

$$\sigma^2_{large} = \frac{(1 - w_{ms})}{(1 - w_{mb})} \cdot \sigma^2_b \tag{46}$$

$$\tag{47}$$

are still valid, all we need to use is the generalized formulas for w_{ms} and w_{mb}.

The same type of generalization may be extended to the scaling of slope covariances. Except that now we only care for the case in which $L_m > L_s$ ($K_m < K_s$):

$$G_{xx} \sim 0.5 \int_{K_m}^{K_s} K^2 S(K)\, dK = 0.5 \frac{A}{(3 - \beta_1)} \left({K_s}^{(3-\beta_1)} - {K_m}^{(3-\beta_1)} \right) \tag{48}$$

and

$$G^b_{xx} \sim 0.5 \int_{K_m}^{K_b} K^2 S(K)\, dK \tag{49}$$

$$= 0.5 \frac{A}{(3 - \beta_1)} \left[\frac{(3 - \beta_1)}{(3 - \beta_2)} \left(\frac{K_0}{K_b} \right)^{(\beta_2 - \beta_1)} {K_b}^{(3-\beta_1)} - {K_m}^{(3-\beta_1)} \right. \tag{50}$$

$$+ \left. \frac{(\beta_1 - \beta_2)}{(3 - \beta - 1)(3 - \beta_2)} {K_0}^{(3-\beta_1)} \right] \tag{51}$$

which implies that

$$G_{xx} \sim r \cdot G^b_{xx} \tag{52}$$

where the scaling factor now reads

$$r = \frac{1 - b^{(3-\beta_1)}}{\frac{(3-\beta_1)}{(3-\beta_2)} \left(\frac{L_b}{L_0} \right)^{(\beta_2-\beta_1)} \left(\frac{L_s}{L_b} \right)^{(3-\beta_1)} - b^{(3-\beta_1)} + \frac{(\beta_1-\beta_2)}{(3-\beta_1)(3-\beta_2)} \left(\frac{L_s}{L_0} \right)^{(3-\beta_1)}} \tag{53}$$

Examples of scaling factors for a range of model resolutions are shown in Figure 2.

6 Implementation in GenPhysX

Only the formulas based on the 1-exponent spectrum are currently implemented in GenPhysX. To activate this scale-separation method in GenPhysX, it suffices to set the option

```
-subgrid SPLIT
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The default values of the spectrum exponent β and the separation wavelength L_s (in m) are set, respectively, as

```
set Const(beta)      2.
set Const(lres)      5000.0
```

The values of these parameteres may in principle be re-set by the user.

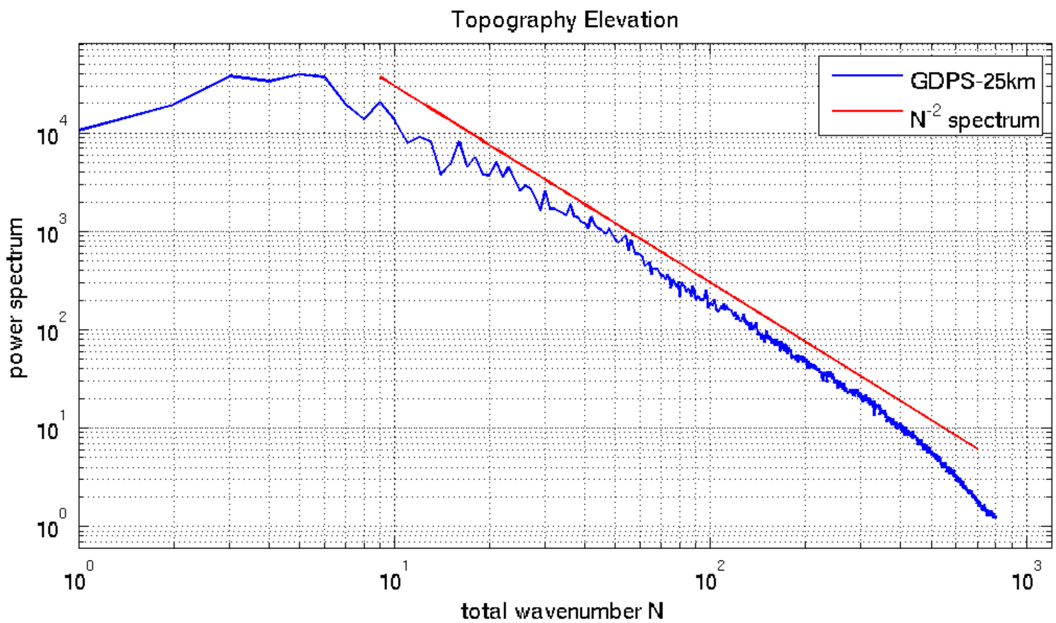

Figure 1: Power spectrum of orography derived from the resolved (non-filtered) topography elevation field of the GDPS-25km model. In the x-axis, N is what was indicated as K in the formulas above. A spectrum with $\beta = 2$ is also shown for reference.

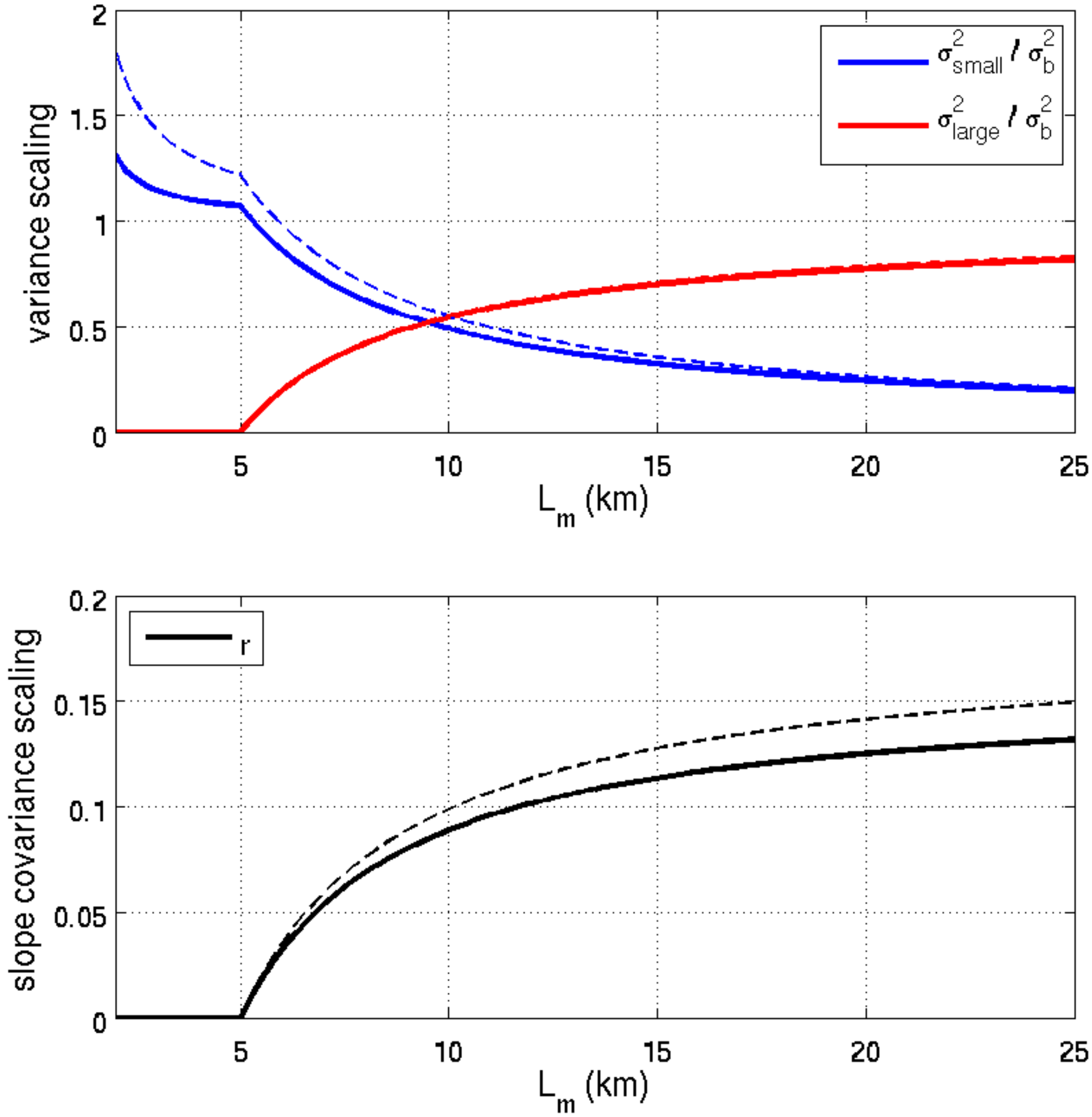

Figure 2: Scaling factors for variances and slope covariances for various model resolutions, assuming a database with resolution $L_b = 0.9km$. Solid lines correspond to the 2-exponent slope assumption $\beta_1 = 1.9, \beta_2 = 2.8$, and dashed lines correspond to the single slope assumption ($\beta_1 = \beta_2 = 2$).